Tripoli university
Faculty of engineering
EE department
EE 313 chapter4 tutorial#1

Problem#1

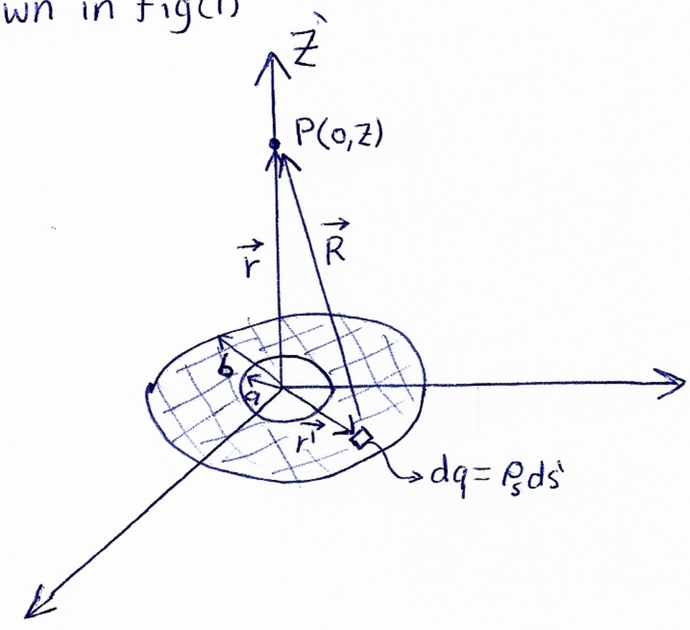
An annular disk of charge lies in the $z=0$ plane centered at the origin, with the uniform charge density ρ_s between its inner and outer radii a and b.

Find the electric field on z-axis from:-

(i) Direct integration.
(ii) gradient of the electric potential.

Solution:

The disk is shown in fig(1)



Fig(1).

where :

$\vec{r}$ is the position vector of the field point $P(0,z)$.

$\vec{r'}$ is the position vector of the charge point.

$\vec{R}$ is the vector from the charge point to the field point.

From the difinition of vector addition: $\vec{R} = \vec{r} - \vec{r'}$, where :-

$\vec{r} = z\vec{a_z}$, $\vec{r'} = \rho'\vec{a_\rho}$

where primes always given to the coordinates of the source point.

$$\therefore \vec{R} = -\rho'\vec{a_\rho} + z\vec{a_z} \Rightarrow \vec{a_R} = \frac{\vec{R}}{|R|} = \frac{-\rho'\vec{a_\rho} + z\vec{a_z}}{\sqrt{(\rho')^2 + z^2}}$$

$$\vec{E} = \int_s \frac{\rho_s \, ds'}{4\pi\epsilon_0 |R|^2}\vec{a_R} = \int_s \frac{\rho_s}{4\pi\epsilon_0 \left((\rho')^2 + z^2\right)} \frac{-\rho'\vec{a_\rho} + z\vec{a_z}}{\sqrt{(\rho')^2 + z^2}} \, ds'$$

$$= \int_{\phi'=0}^{2\pi} \int_{\rho'=a}^{b} \frac{\rho_s}{4\pi\epsilon_0} \frac{-\rho'\vec{a_\rho} + z\vec{a_z}}{\left((\rho')^2 + z^2\right)^{3/2}} \rho' d\rho' d\phi'$$

$$= \frac{\rho_s}{4\pi\epsilon_0} \left[-\int_{\phi'=0}^{2\pi} \int_{\rho'=a}^{b} \frac{(\rho')^2 \vec{a_\rho}}{\left((\rho')^2 + z^2\right)^{3/2}} d\rho' d\phi' + z\vec{a_z} \int_{\phi'=0}^{2\pi} \int_{\rho'=a} \frac{\rho' d\rho' d\phi'}{\left((\rho')^2 + z^2\right)^{3/2}} \right]$$

where $\vec{a_\rho}$ cannot be out of the integration because it is a function of ϕ'. However this integration must be zero since by symmetry, E has no $\vec{a_\rho}$ component. Hence,

$$\vec{E} = \left[2\pi z \vec{a_z} \int_a^b \frac{\rho' d\rho'}{\left((\rho')^2 + z^2\right)^{3/2}} \right] \frac{\rho_s}{4\pi\epsilon_0}$$

To evaluate this integration, let $\rho' = z\tan\alpha \Rightarrow$

$d\rho' = z\sec^2\alpha \, d\alpha$:

$$I = \int \frac{\rho' d\rho'}{((\rho')^2+z^2)^{3/2}} = \int \frac{z^2 \tan\alpha \sec^2\alpha}{(z^2\tan^2\alpha + z^2)^{3/2}} d\alpha$$

and by using the identity $\tan^2 x + 1 = \sec^2 x$:

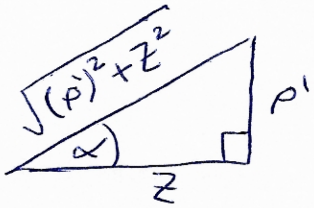
$$I = \int \frac{z^2 \tan\alpha \sec^2\alpha}{(z^2\sec^2\alpha)^{3/2}} d\alpha = \int \frac{z^2 \tan\alpha \sec^2\alpha}{z^3 \sec^3\alpha} d\alpha$$

$$= \int \frac{\tan\alpha}{z\sec\alpha} d\alpha = \frac{1}{z}\int \sin\alpha \, d\alpha$$

where $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$, $\sec\alpha = \frac{1}{\cos\alpha}$.

$$I = \frac{-1}{z}\cos\alpha$$

$$= \frac{-1}{\sqrt{(\rho')^2+z^2}}$$



$$\cancel{\therefore \vec{E} = 2\pi z \vec{a}_z}$$

$$\therefore \vec{E} = \frac{\rho_s}{4\pi\epsilon_0}\left[2\pi z \vec{a}_z \left. \frac{-1}{\sqrt{(\rho')^2+z^2}} \right|_a^b \right]$$

$$= \vec{a}_z \frac{\rho_s}{2\epsilon_0} \left[\frac{z}{\sqrt{a^2+z^2}} - \frac{z}{\sqrt{b^2+z^2}} \right]$$

(ii)

$$\phi = \int_S \frac{\rho_s \, ds'}{4\pi\epsilon_0 |R|} = \int_{\phi=0}^{2\pi}\int_{\rho'=a}^{b} \frac{\rho_s \, \rho' d\rho' d\phi'}{4\pi\epsilon_0 \sqrt{(\rho')^2 + z^2}}$$

$$= \frac{\rho_s}{4\pi\epsilon_0}(2\pi)\int_a^b \frac{\rho' d\rho'}{\sqrt{(\rho')^2+z^2}}$$

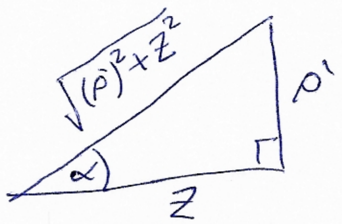
let $\rho' = z\tan\alpha \Rightarrow d\rho' = z\sec^2\alpha \, d\alpha$

$$\phi = \frac{\rho_s}{2\epsilon_0}\int \frac{z^2 \tan\alpha \sec^2\alpha \, d\alpha}{\sqrt{z^2(\tan^2\alpha + 1)}} = \frac{\rho_s}{2\epsilon_0}\int \frac{z^2 \tan\alpha \sec^2\alpha \, d\alpha}{z\sec\alpha}$$

$$= \frac{\rho_s}{2\epsilon_0} z \int \tan\alpha \sec\alpha \, d\alpha = \frac{z\rho_s}{2\epsilon_0}\sec\alpha$$

$$= \frac{z\rho_s}{2\epsilon_0}\left(\frac{\sqrt{(\rho')^2+z^2}}{z}\right)\Bigg|_a^b$$

$$= \frac{\rho_s}{2\epsilon_0}\left(\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right)$$



$$\vec{E} = -\vec{\nabla}\phi = -\vec{a}_\rho \frac{\partial\phi}{\partial\rho} - \frac{1}{\rho}\frac{\partial\phi}{\partial\phi} - \vec{a}_z\frac{\partial\phi}{\partial z}$$

$$= -\vec{a}_z\left[\frac{\rho_s}{2\epsilon_0}\left(\frac{2z}{2\sqrt{b^2+z^2}} - \frac{2z}{2\sqrt{a^2+z^2}}\right)\right]$$

$$= \vec{a}_z \frac{\rho_s}{2\epsilon_0}\left(\frac{z}{\sqrt{a^2+z^2}} - \frac{z}{\sqrt{b^2+z^2}}\right)$$

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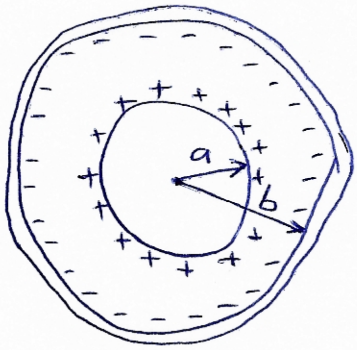
~~Fall 2012~~

Problem #2

Use Gauss's law and symmetry to derive the expression for D(r) of the spherical capacitor of fig. shown. Express the potential $\phi(r)$ at any location between the conductors, using the negative conductor $(r=b)$ as the potential reference. Infer from this the total voltage V between the conducting spheres and find the capacitance



Solution

Choosing Gauss surface as a sphere with radius $a<r<b$:

$$\int_S \bar{D} \cdot \bar{ds} = Q$$

$$D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi = Q$$

$$4\pi r^2 D_r = Q$$

$$\therefore \vec{D} = \vec{a}_r \frac{Q}{4\pi r^2} \quad \Rightarrow \vec{E} = \vec{a}_r \frac{Q}{4\pi \epsilon_o r^2}$$

$$\Phi(r) = -\int_b^r \vec{a}_r \frac{Q}{4\pi\epsilon_0 r^2} \cdot dr\,\vec{a}_r$$

$$= -\frac{Q}{4\pi\epsilon_0}\int_b^r \frac{dr}{r^2}$$

$$= -\frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_b^r = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r}-\frac{1}{b}\right]$$

$$V = \Phi(a) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right]$$

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right]} = \frac{4\pi\epsilon_0}{\frac{1}{a}-\frac{1}{b}}$$

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